Estimates of eigenvalues of Schrödinger operators on the half-line with complex-valued potentials

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Abstract

Estimates for eigenvalues of Schrödinger operators on the half-line with complex-valued potentials are established. Schrödinger operators with potentials belonging to weak Lebesque's classes are also considered. The results cover those known previously due to R. L. Frank, A. Laptev and R. Seiringer [In spectral theory and analysis, vol. 214, Oper. Theory Adv. Appl., pag. 39-44; Birkhäuser/Springer Basel.]

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1 Introduction

This paper is motivated by the recent work [FLS11] in which estimates for non-positive eigenvalues of Schrödinger operators on the half-line are given. The estimates obtained in [FLS11] revised a well-known result, it is mentioned in [Kel61], according to which any negative eigenvalue λ of the (self-adjoint) Schrödinger operator $H(= -d^2/\,dx^2 + q(x))$ satisfies

$$|\lambda|^{1/2} \leq \frac{1}{2}\int_{-\infty}^{\infty} |q(x)|\,dx. \tag{1.1}$$

This result, as was pointed out in [AAD01], remains valid for the case of non-self-adjoint Schrödinger operators as well. In [AAD01] it is proved (1.1) provided that the potential q, being in general a complex-valued function, belongs to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. The mentioned paper [FLS11] concerns Schrödinger operator on $L_2(\mathbb{R}_+)$ assigned with Dirichlet (or also Neumann) boundary conditions. Assuming that q is a summable (in general, complex-valued) function, in [FLS11] instead of (1.1) it is proved that for any (non-positive) eigenvalue $\lambda = |\lambda|e^{i\theta}$

$(0 < \theta < 2\pi)$ of the Schrödinger operator H in $L_2(\mathbb{R}_+)$ (for instance, with Dirichlet boundary condition) satisfies

$$|\lambda|^{1/2} \le \frac{1}{2} g(\cot(\theta/2)) \int_0^\infty |q(x)|\, dx, \tag{1.2}$$

where $g(t) := \sup_{y \ge 0} |e^{ity} - e^{-y}|$.

In contrast with [FLS11] we study the problem for the general case of the Schrödinger operator H considered acting in the Banach space $L_p(\mathbb{R}_+)(1 < p < \infty)$ by assuming that the potential q admits a factorization $q = ab$, where $a \in L_r(\mathbb{R}_+)$ and $b \in L_s(\mathbb{R}_+)$ for some $r, s, 0 < r, s \le \infty$. Under additional subordinate type conditions on the potential q, in order to guarantee the relatively compactness of q viewed as a perturbation of $-d^2/dx^2$, the Schrödinger operator H is defined as a natural closed extension of $-d^2/dx^2 + q$ in $L_p(\mathbb{R}_+)$ having its essential spectrum (as the unperturbed operator) the semi-axis $\mathbb{R}_+$. In this framework the problem reduces to estimation of resolvent of the unperturbed operator bordered by adequate operators of multiplications. First we analyze the situation of Dirichlet boundary conditions, and then we show that the arguments applied are available for the general case when mixed boundary conditions (in particular, for Neumann boundary conditions) are imposed. We prove that if $0 < r \le \infty, p \le s \le \infty, r^{-1} + s^{-1} < 1$, then for eigenvalues λ lying out of $\mathbb{R}_+$ (the essential spectrum) there holds

$$|\lambda|^{1+\alpha} \le (\alpha \sin(\theta/2))^{-2} \|a\|_r^{2\alpha} \|b\|_r^{2\alpha}, \tag{1.3}$$

where $\alpha := (1 - r^{-1} - s^{-1})^{-1}$; as above $\theta = arg\lambda \quad (0 < \theta < 2\pi)$. In the extremal case $\alpha = \infty$ (i.e., $r^{-1} + s^{-1} = 1$) the following estimate

$$|\lambda|^{1/2} \le \frac{1}{2} g(\cot(\theta/2)) \|a\|_p \|b\|_{p'} \tag{1.4}$$

holds true, where g is determined as in (1.2); $p^{'} (= p/(p-1))$ denotes for the conjugate exponent of p. Clearly, if a, b are taken as $|a| = |b| = |q|^{1/2}$, for the case $p = 2$, (1.4) leads to (1.2) that, as was already mentioned, is due to R.L.Frank, A.Laptev and R.Seiringer [FLS11].

Diverse estimates useful in applications can be derived from the general results mentioned above. So, letting $q = L_{\gamma+1/2}(\mathbb{R}_+)$ for $\gamma > 1/2$ if $1 < p \le 2$ and $2\gamma > p - 1$ if $p > 2$, the eigenvalues $\lambda = |\lambda| e^{i\theta} (0 < \theta < 2\pi)$ of H are confined according to the following estimate

$$|\lambda|^\gamma \le \left(\frac{2\gamma + 1}{2\gamma - 1} \sin \frac{\theta}{2} \right)^{1/2 - \gamma} \int_0^\infty |q(x)|^{\gamma + 1/2}\, dx. \tag{1.5}$$

For the case of self-adjoint Schrödinger operators considered on the whole line a similar inequality to (1.5) was pointed out by E.H.Lieb and W.Thirring [LT76] (cf. also the discussion undertaken in this context in [FLS11]; see Remark 1.6 [FLS11]).

Estimates involving decaying potentials can also be derived directly from the general results. So, if it is taken $a(x) = (1 + x)^{-\tau}$ and $b(x) = (1 + x)^{\tau} q(x)$ by

assuming that $(1+x)^\tau q \in L_r(\mathbb{R}_+)$ with $\tau r > 1$, then the eigenvalues λ (with $\theta = arg\lambda, 0 < \theta < 2\pi$) of H satisfy

$$|\lambda|^{r-1} \leq \frac{1}{\tau r - 1}\left(\frac{r}{r-2}\sin\frac{\theta}{2}\right)^{2-r}\int_0^\infty |(1+x)^r q(x)|^r\, dx.$$

It stands to reason that other weight functions like, for instance, $e^{\tau|x|^\alpha}$ with $\tau > 0, \alpha \in \mathbb{R}$, can be also involved.

Finally, note that the arguments by interplaying with interpolation methods [BL76] extend the obtained results to more general case of Schrödinger operators with potentials belonging to weak Lebesgue's spaces. A version of (1.5) for this case is the following one

$$|\lambda|^\gamma \leq C \sup_{t>0}(t^{\gamma+1/2}\lambda_q(t))$$

with γ as in (1.5); λ_q denotes the distribution function of the potential q with respect to the standard Lebesgue measure on $\mathbb{R}_+$.

The paper is organized as follows. In Section 2 the problem is discussed for Schrödinger operators with Lebesgue power-summable potentials. Section 2 is divided in two subsections. The first is concerned with Schrödinger operators with the Dirichlet boundary conditions. In the second one we discuss the general situation when the mixed boundary conditions are imposed. In Section 3 we treat the case of potentials belonging to weak Lebesgue's type spaces.

2 Lebesgue summable type potentials

We consider the Schrödinger operator H defined in the space $L_p(\mathbb{R}_+)(1 < p < \infty)$ as a closed extension of the formal differential operator $-\,d^2/\,dx^2 + q(x)$. For it should be posed suitable conditions on the potential q (in an averaged sense to be small at infinity) ensuring the relatively compactness of q regarded as a perturbation operator. We assume that q admits a factorization $q = ab$ with $a \in L_r(\mathbb{R}_+)$ and $b \in L_s(\mathbb{R}_+)$ for some $0 < r, s \leq \infty$. Further conditions on the potential q under which the main results are obtained ensured that the essential spectrum of H is the same in each of Banach space $L_p(\mathbb{R}_+)$ with $1 < p < \infty$, and filling the semi-axis $\mathbb{R}_+$. To this end we restrict ourselves to refer [Sch71] for details and other diverse related conditions concerning general elliptic differential operators. In this framework the problem of evaluation for eigenvalues (lying outside of the essential spectrum) of H reduces to norm estimation of the resolvent of the unperturbed operator bordered by adequate operators of multiplication as is described below.

1. We first consider Dirichlet boundary condition case. The unperturbed operator $H_0 = -\,d^2/\,dx^2$ is taken with the domain the Sobolev space $W_p^2(\mathbb{R}_+)$ consisting of all functions $u \in L_p(\mathbb{R}_+)$ such that u, u' are absolutely continuous with $u'' \in L_p(\mathbb{R}_+)$ and $u(0) = 0$. The operator H_0 is closed and $\sigma(H_0) = [0, \infty)$. For any $\lambda \in \mathbb{C} \setminus [0, \infty)$ the resolvent $R(\lambda; H_0) = (H_0 - \lambda I)^{-1}$ is the integral operator

$$R(\lambda; H_0)v(x) = -\frac{1}{2i\mu}\int_0^\infty e^{i\mu|x-y|}v(y)\,dy + \frac{1}{2i\mu}\int_0^\infty e^{i\mu(x+y)}v(y)\,dy, \quad (2.1)$$

where $\mu = \lambda^{1/2}$ is chosen so that $\operatorname{Im}\mu > 0$.

We denote by A, B the operators of multiplication by a, b, respectively, and evaluate the norm of the bordered resolvent $BR(\lambda; H_0)A$. For we choose $\beta > 0$ and $\gamma > 0$ such that the evaluations

$$\|au\|_\beta \le \|a\|_r \|u\|_p, \quad \beta^{-1} = r^{-1} + p^{-1}, \tag{2.2}$$

and

$$\|bv\|_p \le \|b\|_s \|v\|_\gamma, \quad p^{-1} = s^{-1} + \gamma^{-1} \tag{2.3}$$

hold true (those can be obtained by the use of Hölder inequality). In this way, the operator is bounded viewed as an operator acting from $L_p(\mathbb{R}_+)$ to $L_\beta(\mathbb{R}_+)$ and, respectively, B as a bounded operator from $L_\gamma(\mathbb{R}_+)$ to $L_p(\mathbb{R}_+)$.

Next, we let

$$k(x, y; \lambda) = -\frac{1}{2i\mu}(e^{i\mu|x-y|} - e^{i\mu(x+y)}), \quad 0 < x, y < \infty,$$

for the kernel of the resolvent $R(\lambda; H_0)$ and proceed as follows. First, we take an α, $1 \le \alpha < \infty$, and observe that

$$\sup_{0<x<\infty} \|k(x, \cdot; \lambda)\|_\alpha \le 1/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha}.$$

In fact, for any x, $0 < x < \infty$, we have

$$\int_0^\infty |e^{i\mu|x-y|}|^\alpha \, dy = \int_0^x e^{-\alpha(\operatorname{Im}\mu)(x-y)} \, dy + \int_x^\infty e^{-\alpha(\operatorname{Im}\mu)(-x+y)} \, dy =$$

$$= \frac{1}{\alpha \operatorname{Im}\mu}\left(2 - e^{-\alpha(\operatorname{Im}\mu)x}\right),$$

and

$$\int_0^\infty |e^{i\mu(x+y)}|^\alpha \, dy = \int_0^\infty e^{-\alpha(\operatorname{Im}\mu)(x+y)} \, dy = \frac{1}{\alpha \operatorname{Im}\mu} e^{-\alpha(\operatorname{Im}\mu)x},$$

hence

$$\|k(x, \cdot; \lambda)\|_\alpha = \left(\int_0^\infty \left|\frac{1}{2i\mu}\left(e^{i\mu|x-y|} - e^{i\mu(x+y)}\right)\right|^\alpha dy\right)^{1/\alpha} \le$$

$$\le \frac{1}{2|\mu|}\left(\left(\int_0^\infty |e^{i\mu|x-y|}|^\alpha \, dy\right)^{1/\alpha} + \left(\int_0^\infty |e^{i\mu(x+y)}|^\alpha \, dy\right)^{1/\alpha}\right) =$$

$$= \frac{1}{2|\mu|}\left(\left(\frac{1}{2 \operatorname{Im}\mu}(2 - e^{-\alpha(\operatorname{Im}\mu)x})\right)^{1/\alpha} + \left(\frac{1}{2 \operatorname{Im}\mu} e^{-\alpha(\operatorname{Im}\mu)x}\right)^{1/\alpha}\right) =$$

$$= \frac{1}{2|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha}}\left((2 - e^{-\alpha(\operatorname{Im}\mu)x})^{1/\alpha} + e^{-(\operatorname{Im}\mu)x}\right),$$

i.e.,

$$\|k(x,\cdot;\lambda)\|_\alpha \le \frac{1}{2|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha}} \left((2-e^{-\alpha(\operatorname{Im}\mu)x})^{1/\alpha} + e^{-(\operatorname{Im}\mu)x}\right).$$

The optimal value of the right member for varying x, $0 < x < \infty$, is equal to $1/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha}$, and, thus, the desired inequality follows.

By Minkowski's inequality, it follows

$$\|R(\lambda;H_0)v\|_\alpha = \left(\int_0^\infty \left|\int_0^\infty k(x,y;\lambda)v(y)\,dy\right|^\alpha dx\right)^{1/\alpha} \le$$

$$\le \int_0^\infty \left(\int_0^\infty |k(x,y;\lambda)|^\alpha\,dx\right)^{1/\alpha} |v(y)|\,dy \le \sup_{0<y<\infty} \|k(\cdot,y;\lambda)\|_\alpha \|v\|_1,$$

and, since the variables in the kernel $k(x,y;\lambda)$ are equal right, one has

$$\|R(\lambda;H_0)v\|_\alpha \le (1/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha})\|v\|_1. \tag{2.4}$$

On the other hand, by Hölder's inequality, there holds

$$|R(\lambda;H_0)v(x)| = \left|\int_0^\infty k(x,y;\lambda)v(y)\,dy\right| \le$$

$$\left(\int_0^\infty |k(x,y;\lambda)|^\alpha\,dy\right)^{1/\alpha} \left(\int_0^\infty |v(y)|^{\alpha'}\,dy\right)^{1/\alpha'} = \|k(x,\cdot;\lambda)\|_\alpha \|v\|_{\alpha'},$$

that yields that

$$\|R(\lambda;H_0)v\|_\infty \le (1/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha})\|v\|_{\alpha'}. \tag{2.5}$$

The evaluation (2.4) means that the resolvent operator $R(\lambda;H_0)$ is bounded regarded as an operator from $L_1(\mathbb{R}_+)$ to $L_\alpha(\mathbb{R}_+)$ while (2.5) means the boundedness of $R(\lambda;H_0)$ as an operator from $L_{\alpha'}(\mathbb{R}_+)$ to $L_\infty(\mathbb{R}_+)$. In both cases its norm is bounded by $1/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha}$. By applying the Riesz-Thorin interpolation theorem (see, for instance, [BL76]; Theorem 1.1.1) we conclude that the resolvent operator $R(\lambda;H_0)$ is bounded from $L_\beta(\mathbb{R}_+)$ to $L_\gamma(\mathbb{R}_+)$ provided that

$$\frac{1}{\beta} = \frac{1-\theta}{1} + \frac{\theta}{\alpha'}, \quad \frac{1}{\gamma} = \frac{1-\theta}{\alpha} + \frac{0}{\infty}, \quad 0<\theta<1.$$

Moreover, the corresponding value of its norm does not exceed $1/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha}$. Eliminating θ, we find

$$\alpha^{-1} + \beta^{-1} = \gamma^{-1} + 1,$$

which, in view of restriction in (2.2) and (2.3), implies $\alpha = (1 - r^{-1} - s^{-1})^{-1}$. Note that due to the fact that $1 \le \alpha < \infty$ it must be $0 \le r^{-1} + s^{-1} < 1$. In these conditions we obtain the following estimate

$$\|BR(\lambda;H_0)Au\|_p \le (1/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha})\|a\|_r\|b\|_s\|u\|_p,$$

and, therefore, for any eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ of H it should be fulfilled

$$|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha} \le \|a\|_r\|b\|_s,$$

that, by letting $\lambda = |\lambda|e^{i\theta}$, $0 < \theta < 2\pi$, provides to the following estimate

$$|\lambda|^{1+\alpha} \le (\alpha \sin(\theta/2))^{-2} \|a\|_r^{2\alpha} \|b\|_s^{2\alpha}. \tag{2.6}$$

We have proved the following result.

Theorem 2.1. *Let $1 < p < \infty$, $0 < r \le \infty$, $p \le s \le \infty$, $r^{-1} + s^{-1} < 1$, and assume $q = ab$, where $a \in L_r(\mathbb{R}_+)$ and $b \in L_s(\mathbb{R}_+)$. Then, for any eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ of the operator H, considered acting in $L_p(\mathbb{R}_+)$, there holds* (2.6)*.*

Diverse estimates useful in applications can be derived from the above general result. If in Theorem 2.1 is taken $r = s$, there obtains the following result.

Corollary 2.2. *Suppose $q = ab$, where $a, b \in L_r(\mathbb{R}_+)$ with $r > 2$ if $1 < p \le 2$ and $p \le r \le \infty$ if $p > 2$. Then for any eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ of H, considered acting in $L_p(\mathbb{R}_+)$, there holds*

$$|\lambda|^{r-1} \le \left(\frac{r}{r-2} \sin \frac{\theta}{2} \right)^{2-r} \|a\|_r^r \|b\|_r^r. \tag{2.7}$$

The following particular case presents peculiar interest in many situations.

Corollary 2.3. *Let $\gamma > 1/2$ if $1 < p \le 2$ and $2\gamma \ge p - 1$ if $p > 2$, and suppose*

$$q \in L_{\gamma+1/2}(\mathbb{R}_+).$$

Then any eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ of the operator H in $L_p(\mathbb{R}_+)$ satisfies

$$|\lambda|^{\gamma} \le \left(\frac{2\gamma+1}{2\gamma-1} \sin \frac{\theta}{2} \right)^{1/2-\gamma} \int_0^\infty |q(x)|^{\gamma+1/2}\, dx. \tag{2.8}$$

Proof. In Corollary 2.2 it suffices to let $r = 2\gamma + 1$ and take $a(x) = |q(x)|^{1/2}$ and $b(x) = (sgnq(x))|q(x)|^{1/2}$, where $sgnq(x) = q(x)/|q(x)|$ of $q(x) \ne 0$ and $sgnq(x) = 0$ if $q(x) = 0$. □

Remark 2.4. For the self-adjoint case can be occurred only negative eigenvalues of H, and thus (2.8) becomes

$$|\lambda|^{\gamma} \le \left(\frac{2\gamma-1}{2\gamma+1} \right)^{\gamma-1/2} \int_0^\infty |q(x)|^{\gamma+1/2}\, dx. \tag{2.9}$$

Similar estimates for whole-line operators were pointed out in [Kel61] or [LT76]. For related results and discussion in other contexts see also [DN02], [FLLS06], [FLS11] and [LS09].

Estimates involving decaying potentials can be also obtained directly from the general results. So, if we take in (2.7) $a(x) = (1+x)^{-\tau}$ and $b(x) = (1+x)^{\tau} q(x)$, we obtain the following result.

Corollary 2.5. *Suppose $(1+x)^{\tau} q \in L_r(\mathbb{R}_+)$ with $\tau r > 1$ and r as in Corollary 2.2. Then any eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ of the operator H, considered acting in $L_p(\mathbb{R}_+)$, satisfies*

$$|\lambda|^{r-1} \le \frac{1}{\tau r - 1} \left(\frac{r}{r-2} \sin \frac{\theta}{2} \right)^{2-r} \int_0^\infty |(1+x)^{\tau} q(x)|^r\, dx. \tag{2.10}$$

The following is also a simple consequence of above general results.

Corollary 2.6. *Let r and p be as in Corollary 2.2, and suppose $e^{\tau x}q \in L_r(\mathbb{R}_+)$ for $\tau > 0$. Then, any eigenvalue $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$ of the operator H, considered acting in $L_p(\mathbb{R}_+)$ satisfies*

$$|\lambda|^{r-1} \leq \frac{1}{\tau r}\left(\frac{r}{r-2}\sin\frac{\theta}{2}\right)^{2-r}\int_0^\infty e^{\tau r x}|q(x)|^r\,dx. \tag{2.11}$$

For the extremal case $\alpha = \infty$ we have

$$\sup_{0<x<\infty}\|k(x,\cdot;\lambda)\|_\infty = \sup_{0<x,y<\infty}\frac{1}{2|\mu|}|e^{i\mu|x-y|} - e^{i\mu(x+y)}|.$$

This supremum has been computed in [FLS11] (see [FLS11], proof of Theorem 1.1, and also Lemma 1.3). It turns out that

$$\sup_{0<x,y<\infty}|e^{i\mu|x-y|} - e^{i\mu(x+y)}| = g(\cot(\theta/2)), \quad \theta = \arg\lambda,$$

where

$$g(a) := \sup_{0<x,y<\infty}|e^{iay} - e^{-y}|$$

(g is an even function: $g(-a) = g(a)$). Let us show this fact for the sake of completeness. It follows from the following simple relations:

$$\sup_{0<x,y<\infty}|e^{i\mu|x-y|} - e^{i\mu(x+y)}| = \sup_{0<y<x}|e^{i\mu(x-y)} - e^{i\mu(x+y)}| =$$

$$= \sup_{y>0}|1 - e^{2i\mu y}| = \sup_{y>0}|e^{-i(\cot(\theta/2))y} - e^{-y}| = g(\cot(\theta/2)).$$

Thus,

$$\sup\|k(x,\cdot;\lambda)\|_\infty = \frac{1}{2|\mu|}g(\cot(\theta/2)),$$

and, therefore, the resolvent operator $R(\lambda; H_0)$ is bounded from $L_1(\mathbb{R}_+)$ to $L_\infty(\mathbb{R}_+)$, and

$$\|R(\lambda; H_0)v\|_\infty \leq \frac{1}{2|\mu|}g(\cot(\theta/2))\|v\|_1. \tag{2.12}$$

In this case it should be taken $\beta = 1$, $\gamma = \infty$, then (2.2) and (2.3) held for $r = p^{'}$ and $s = p$ that together with (2.12) implies

$$\|BR(\lambda; H_0)Au\|_p \leq \frac{1}{2|\mu|}g(\cot(\theta/2))\|a\|_{p'}\|b\|_p\|u\|_p.$$

Therefore, for any eigenvalue $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$ of the operator H_0, it should be held (changed a with b)

$$|\lambda|^{1/2} \leq \frac{1}{2}g(\cot(\theta/2))\|a\|_p\|b\|_{p'}. \tag{2.13}$$

Thus, there holds the following result.

Theorem 2.7. *Let $1 < p < \infty$, and let $q = ab$ with $a \in L_p(\mathbb{R}_+)$ and $b \in L_{p'}(\mathbb{R}_+)$. Then any eigenvalue $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$ of the operator H, considered acting in $L_p(\mathbb{R}_+)$, satisfies* (2.13).

In particular, for the Hilbert space case $p = 2$ there holds

$$|\lambda|^{1/2} \leq \frac{1}{2} g(\cot(\theta/2)) \|a\|_2 \|b\|_2. \tag{2.14}$$

Remark 2.8. If in (2.14) it is taken $a(x) = |q(x)|^{1/2}$ and $b(x) = (sgn q(x))|q(x)|^{1/2}$, then

$$\|a\|_2^2 = \|b\|_2^2 = \int_0^\infty |q(x)|\, dx,$$

and estimate (2.14) becomes

$$|\lambda|^{1/2} \leq \frac{1}{2} g(\cot(\theta/2)) \int_0^\infty |q(x)|\, dx.$$

This result was established in [FLS11] (it is presented in Theorem 1.1 [FLS11] as the main result).

Remark 2.9. From results presented in Theorem 2.7 can be established various special estimates useful for applications. For instance, arguing as in the case of Corollary 2.5, it can be derived the following estimate

$$|\lambda|^{1/2} \leq \frac{1}{2} g(\cot(\theta/2))(p'\tau - 1)^{-1/p'} \|(1+x)^\tau q\|_p$$

provided that $p'\tau > 1$ and $(1+x)^\tau q \in L_p(\mathbb{R}_+)$.

2. By applying the same arguments there can be obtained related estimates for eigenvalues of the operator $H = -\,d^2/\,dx^2 + q(x)$ considered with general boundary conditions like $u'(0) = \sigma u(0)$ ($0 \leq \sigma < \infty$; in case $\sigma = \infty$ it is taken the Dirichlet condition; $\sigma = 0$ corresponds to the Neumann boundary condition $u'(0) = 0$). We attach to this general situation all conventions made above for the Dirichlet boundary condition case concerning the exact definition of the perturbed operator H. In what follows, the corresponding perturbed operator is denoted by H_σ (it will be no confusion with the notation H_0 used as unperturbed operator and the operator H corresponding to the Neumann boundary condition case $\sigma = 0$). In order to apply the arguments used above, we first note that the resolvent operator of the unperturbed operator (with general boundary conditions) is an integral operator with the kernel

$$k_\sigma(x, y; \lambda) = -\frac{1}{2\mu}\left(e^{i\mu|x-y|} - \frac{\sigma + i\mu}{\sigma - i\mu} e^{i\mu(x+y)}\right), \quad 0 < x, y < \infty.$$

Next, we take α, $1 \leq \alpha < \infty$, and observe that

$$\sup_{0<x<\infty} \|k_\sigma(x, \cdot; \lambda)\|_\alpha \leq 1/|\mu|(\alpha \operatorname{Im} \mu)^{1/\alpha}.$$

We take $\operatorname{Im} \mu > 0$ and then $|(\sigma + i\mu)/(\sigma - i\mu)^{-1}| \leq 1$, and arguments similar to that used in proving the estimation for the Dirichlet boundary condition case (when $\sigma = \infty$) are applied.

Thus, as is seen, for the operator H_σ, that is, for the case of general boundary conditions, the result given by Theorem 2.1 remains valid as well. We formulate the corresponding result in a separate theorem.

Theorem 2.10. *Under conditions of Theorem 2.1 for any eigenvalue* $\lambda \in \mathbb{C}\backslash\mathbb{R}_+$ *of the operator* H_σ*, considered acting in* $L_p(\mathbb{R}_+)$*, an estimate like* (2.6) *holds true. In particular, for negative eigenvalues* λ *of* H_σ*, there holds*

$$|\lambda|^{1+\alpha} \le \alpha^{-2}\|a\|_r^{2\alpha}\|b\|_s^{2\alpha}.$$

For the extremal case $\alpha = \infty$ we have

$$\sup_{0<x<\infty} \|k_\sigma(x,\cdot;\lambda)\|_\infty = \frac{1}{2|\mu|} g_\sigma(\cot(\theta/2)),$$

g_σ instead of g, where

$$g_\sigma(a) := \sup_{0<x<\infty} \left| e^{iay} - \frac{\sigma + i\mu}{\sigma - i\mu} e^{-y} \right|, \quad a \in \mathbb{R}.$$

Accordingly, the following result holds true.

Theorem 2.11. *Under the conditions of Theorem 2.7 for any eigenvalue* $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ *of the operator* H_σ*, considered acting in* $L_p(\mathbb{R}_+)$*, there holds*

$$|\lambda|^{1/2} \le \frac{1}{2} g_\sigma(\cot(\theta/2))\|a\|_p\|b\|_p.$$

Remark 2.12. The result given by Theorem 2.11 for the Hilbert space case $p = 2$ and $a(x) = |q(x)|^{1/2}$, $b(x) = (sgnq(x))|q(x)|^{1/2}$ was mentioned in [FLS11] (see Proposition 1.5 [FLS11]).

3 The case of potentials from weak Lebesgue's spaces

Estimates for the perturbed eigenvalues can be obtained under slightly weakened conditions on the potentials. It turns out that it can be involved potentials belonging to weak Lebesgue's spaces. To be more precisely we consider a Schrödinger operator $H = -\,d^2/\,dx^2 + q(x)$, where the potential q is written as a product $q = ab$ with $a \in L_{r,w}(\mathbb{R}_+)$ and $b \in L_{s,w}(\mathbb{R}_+)$ (we will use $L_{r,w}$ to denote the so-called weak L_r-spaces). The operator H will be considered acting in the space $L_p(\mathbb{R}_+)$ $(1 < p < \infty)$ and subjected with the Dirichlet boundary condition (there will no loss of generality in supposing only the Dirichlet boundary condition).

We recall that the weak L_r-space $L_{r,w}(\mathbb{R}_+)$ $(0 < r < \infty)$ is the space consisting of all measurable functions on $\mathbb{R}_+$ such that

$$\|f\|_{r,w} := \sup_{t>0}(t^r\lambda_f(t))^{1/r} < \infty,$$

where λ_f denotes the distribution function of f, namely, $\lambda_f(t) = \mu(\{x \in \mathbb{R}_+ : |f(x)| > t\})$, $0 < t < \infty$ (here μ is the standard Lebesgue measure on $\mathbb{R}_+$). The spaces $L_{r,w}(\mathbb{R}_+)$ are special cases of the more general Lorentz spaces $L_{p,r}(\mathbb{R}_+)$

which will also be needed. $L_{p,r}(\mathbb{R}_+)$ $(0 < p, r < \infty)$ is defined as the space of all measurable functions f on $\mathbb{R}_+$ for which

$$\|f\|_{p,r}^r := \int_0^\infty t^r (\lambda_f(t))^{r/p} \frac{dt}{t} < \infty.$$

Note that $L_{r,r}(\mathbb{R}_+) = L_r(\mathbb{R}_+)$, and it will be convenient to let $L_{\infty,r}(\mathbb{R}_+) = L_\infty(\mathbb{R}_+)$.

As in previous section we let A, B denote the operators of multiplication by a, b, respectively. In view of $a \in L_{r,w}(\mathbb{R}_+)$ and $b \in L_{s,w}(\mathbb{R}_+)$, as was assumed, we can apply a result of O'Neil [O'N63] due to of which there can be chosen $\beta > 0$ and $\gamma > 0$ such that the multiplication operator A to be bounded from $L_{p,p}(\mathbb{R}_+)(= L_p(\mathbb{R}_+))$ to $L_{\beta,p}(\mathbb{R}_+)$ and, respectively, B to be bounded from $L_{\gamma,p}(\mathbb{R}_+)$ to $L_{p,p}(\mathbb{R}_+)$ and, moreover,

$$\|Au\|_{\beta,p} \leq c\|a\|_{r,w}\|u\|_p, \quad \beta^{-1} = r^{-1} + p^{-1}, \tag{3.1}$$

and

$$\|Bv\|_p \leq c\|b\|_{s,w}\|v\|_{\gamma,p}, \quad p^{-1} = s^{-1} + \gamma^{-1}. \tag{3.2}$$

Note that in (3.1) and (3.2) the constants in general are distinct, but depending only on r, p and s, p, respectively.

Next, as was shown, the resolvent operator $R(\lambda; H_0)$ of unperturbed operator H_0 acts as a bounded operator from $L_1(\mathbb{R}_+)$ to $L_\alpha(\mathbb{R}_+)$ and, simultaneously, from $L_{\alpha'}(\mathbb{R}_+)$ to $L_1(\mathbb{R}_+)$ for any α, $1 \leq \alpha < \infty$. Besides, in both cases the bound for the norm of $R(\lambda; H_0)$ does not exceed $1/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha}$. By applying the interpolation functor $K_{\theta,p}$ with $0 < \theta < 1$ (cf., [BL76] or [Tri78]), we obtain that $R(\lambda; H_0)$ acts also as a bounded operator from $L_{\beta,p}(\mathbb{R}_+)$ into $L_{\gamma,p}(\mathbb{R}_+)$ provided that

$$\frac{1}{\beta} = \frac{1-\theta}{1} + \frac{\theta}{\alpha'}, \quad \frac{1}{\gamma} = \frac{1-\theta}{\alpha} + \frac{\theta}{\infty}.$$

Moreover,

$$\|R(\lambda; H_0)v\|_{\gamma,p} \leq \Big(C/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha}\Big)\|v\|_{\beta,p}, \tag{3.3}$$

where C is a positive constant depending only on p, γ and β occurred after interpolation. In view of (3.1), (3.2) and (3.3) we conclude that, under our assumption, there holds

$$\|BR(\lambda; H_0)Au\|_p \leq \Big(C/|\mu|(\alpha \operatorname{Im}\mu)^{1/\alpha}\Big)\|a\|_{r,w}\|b\|_{s,w}\|u\|_p$$

with a positive constant C depending only on p, r and s.

In this manner, we obtain an estimate like (2.6), namely

$$|\lambda|^{1+\alpha} \leq C(\alpha \sin(\theta/2))^{-2}\|a\|_{r,w}^{2\alpha}\|b\|_{s,w}^{2\alpha}, \tag{3.4}$$

under more weaker conditions on the potential q than those required in Theorem 2.1. In (3.4), as in (2.6), $\alpha = (1 - r^{-1} - s^{-1})^{-1}$ with the same restrictions on r and s. We have proved the following result.

Theorem 3.1. *Let p, r, s be as in Theorem 2.1, and suppose $q = ab$, where $a \in L_{r,w}(\mathbb{R}_+)$ and $b \in L_{s,w}(\mathbb{R}_+)$. Then, any eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ of the operator H, considered acting in $L_p(\mathbb{R}_+)$, satisfies* (3.4).

Similarly as in previous section diverse estimates for eigenvalues useful in applications can be derived from general result given by Theorem 3.1. Here we restrict ourselves to remark only a version accommodated for potentials from weak Lebesgue's classes of the result presented in Corollary 2.3.

Corollary 3.2. *Let $2\gamma > 1$ if $1 < p \leq 2$ and $2\gamma > p - 1$ if $p > 2$, and suppose $q \in L_{\gamma+1/2,w}(\mathbb{R}_+)$. Then any eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ of the operator H, considered acting in $L_p(\mathbb{R}_+)$, satisfies*

$$|\lambda|^\gamma \leq C \sup_{t>0} \left(t^{\gamma+1/2} \lambda_q(t) \right), \tag{3.5}$$

where $C = C(p, \gamma, \theta)$ is a positive constant depending only on p, γ and θ ($\theta = \arg \lambda$).

Remark 3.3. In (3.5) the value of C can be controlled.

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